INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 16, 65–81 (1993)

A PROBABILISTIC METHOD FOR FIELD SOLUTIONS IN TWO-PHASE FLOWS

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SUMMARY

Solid-particle motion and related transport phenomena in two-phase flow are fluctuating processes in space and time. A deterministic method can describe only partially the intrinsic physics of these processes. In this paper, the fluctuations of the flow parameters are modelled by considering the spatial correlations, and a probabilistic computational method for two-phase flow is presented. The probabilistic governing equations have been discretized in space using a finite volume method, and then solved by applying the Neumann expansion method. This last method is time efficient, and its convergence can be guaranteed even for large fluctuations. A liquid-solid particle mixture flow in a circular pipe is taken as an example. Computational results illustrate the merit of the probabilistic approach for the prediction of two-phase flow phenomena.

KEY WORDS Probabilistic numerical method Neumann expansion method Fluid flow Solid–liquid two-phase flow Stochastic process

INTRODUCTION

Solid-particle motion and transport processes in two-phase flow are random in space and time, sometimes with an organized structure. However, the usual numerical techniques for two-phase flow are deterministic,[1–4] which only partially can describe the complex flow mechanisms. A numerical simulation of particulate two-phase flow within a stochastic framework is the object of this study.

It is well-known that turbulent flow parameters, particularly in two-phase flow, are characterized by probability distribution functions which are spatially correlated. A probabilistic approach accounts at least partially for this intrinsic flow characteristic. The spatial correlations between fluctuations are particularly important in flows with large mixing lengths and organized structures. The difference between the probabilistic and deterministic solutions is more significant if the governing equations are non-linear, and the standard deviations and spatial correlations of the fluctuations are larger.

The uncertainty associated with the spatial variability of particle location, particle size and heterogeneity produces an uncertainty in the flow parameters such as concentration and mixture velocity. In the probabilistic approach discussed here, an input parameter is considered to be a random variable with an associated probability density function at each point in the flow domain. The spatial dependence between neighbouring values of the random variables are defined in terms of a stochastic process model that defines the spatial correlation throughout the system. Related investigations on digital generation of sample functions in multidimensional, multivariable stochastic fields have been carried out by Shinozuka and

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0271–2091/93/010065–17$13.50

Received 1 September 1991
Revised 7 May 1992

Dasgupta,[5] Shinozuka and Deodatis,[6] Vanmarcke,[8,9] Smith and Freeze,[10,11] Bakr *et al.*[12] and Roco *et al.*[13]

It is known that the direct Monte Carlo simulation is prohibitive in many applications, including ours, because of the required computer time. A frequent approximation to direct Monte Carlo simulation is the perturbation method.[5,6] However, this method requires that the variability of the flow parameters to be small. In addition, the method suffers on its accuracy, convergence and computational efficiency in higher-order solutions. These drawbacks are crucial in two-phase flow simulations where, in most of the cases, the variability of fluctuation of the random flow parameter is relatively large and higher order-solutions are required.

The Neumann expansion technique was adopted in the present work within the framework of Monte Carlo method. This technique allows for a large variability of fluctuations of the flow parameters. The convergence of the solution can be guaranteed. The Neumann expansion method has been tested in solid mechanics, for the structural response variability resulting from the spatial variability of material properties.[5,6] Other probabilistic numerical methods to solve field problem has been investigated in soil mechanics[8,9] and ground-water flow and seepage problems.[10,11]

The deterministic finite volume method has been previously applied for two-phase flow in pipes,[4,7] and it will used in the present investigation as reference for comparison. The method is briefly described in Appendix II. The probabilistic numerical approach can be applied also in conjunction with other deterministic numerical method, such as finite element method and finite differences method. The governing equations used here for the one-equation turbulence model of incompressible mixture flow were presented in References 4 and 14. Their application is illustrated with the one-equation eddy-viscosity model because reference deterministic results were available from previous work. The eddy viscosity may be considered as the product of the local turbulence length scale and velocity fluctuation. Since the turbulence length scale was assumed deterministic in the present study, the origin of the eddy-viscosity fluctuations are the velocity fluctuations.

The objective of this paper is to develop a probabilistic numerical algorithm with potential applications to single-fluid and two-phase flow simulations. With several simplifications, the computational algorithm is illustrated for a slurry-pipe flow. Preliminary results obtained with the probabilistic approach are compared with results from the deterministic approach.

MODELLING OF STOCHASTIC FIELDS

The spatial variation of a flow quantity ζ, such as concentration, velocity or eddy viscosity, is assumed to be a homogeneous stochastic process

$$\zeta = \bar{\zeta} + Y, \tag{1}$$

where $\bar{\zeta}$ is the mean and Y is its fluctuation, with mean (expectation) of Y equal to zero

$$E[Y] = 0. \tag{2}$$

Two methods can be used to model the stochastic process Y: (a) the correlation function model and (b) the autoregressive model.

The correlation function model

The usual way of describing a stochastic process Y, which has a spatial dependence, is through the correlation function R between two points

$$R(\Delta \mathbf{r}) = E[Y(\mathbf{r}) \cdot Y(\mathbf{r} + \Delta \mathbf{r})], \tag{3}$$

where, in a two-dimensional domain (x, z), $\mathbf{r}=[x, z]^{\mathrm{T}}$ is the position vector, and $\Delta\mathbf{r}=[\Delta x, \Delta z]^{\mathrm{T}}$ is the separation vector between two points $\mathbf{r}$ and $\mathbf{r}+\Delta\mathbf{r}$.

Let η_x and η_z represent the integral scales of the process[9] in the x and z directions, respectively. They describe how fast the correlation R decreases with Δx and Δz, respectively. The correlation function between two different points, (x, z) and $(x+\Delta x, z+\Delta z)$, can be expressed by

$$R(\Delta x, \Delta z)=\sigma_Y^2 \exp\left\{-\left[\left(\frac{\Delta x}{\eta_x}\right)^2+\left(\frac{\Delta z}{\eta_z}\right)^2\right]\right\}, \tag{4}$$

where σ_Y is the standard deviation of the stochastic field Y. For simplicity, in the present analysis, Y is assumed homogeneous in the (x, z) domain.

If the randomness of the spatial variation is isotropic, then $\eta_x=\eta_z=\eta$, and the correlation function of the spatial variation will depend only on the distance $\Delta r=(\Delta x^2+\Delta z^2)^{0.5}$ between the two points of interest

$$R(\Delta x, \Delta z)=\sigma_Y^2 \exp\left[-\left(\frac{\Delta r}{\eta}\right)^2\right]. \tag{5}$$

Let us consider two areas A_1 and A_2 in a two-dimensional flow domain, where Y is defined. The areas A_1, A_2, A_0, A_{01} $(A_0 \cup A_1)$, A_{02} $(A_0 \cup A_2)$ and A_{012} $(A_0 \cup A_1 \cup A_2)$ are marked in Figure 1. A useful expression for the correlation function $R(A_1, A_2)$ was proposed by Vanmarcke[9]

$$R(A_1, A_2)=\sigma_Y^2 [A_0^2\, \gamma(A_0)-A_{01}^2\, \gamma(A_{01})-A_{02}^2\, \gamma(A_{02})+A_{012}^2\, \gamma(A_{012})], \tag{6}$$

where $\gamma(A)$ is the spatial two-dimensional variance function. As discussed by Vanmarcke, $\gamma(A)$ can be replaced by the product of two one-dimensional variance functions

$$\gamma(A)=\gamma(\Delta r, \Delta l)=\gamma(\Delta r)\cdot\gamma(\Delta l), \tag{7}$$

where Δl is the circumferential space interval. Here, $\gamma(\Delta l)$ [or $\gamma(\Delta r)$] is the one-dimensional variance function

$$\begin{aligned}\gamma(\Delta l)&=1-\frac{\Delta l}{3\eta} && \text{if } \Delta l \leqslant \eta,\\ &=\frac{\eta}{\Delta l}\left(1-\frac{\eta}{3\cdot\Delta l}\right) && \text{if } \Delta l > \eta\end{aligned} \tag{8}$$

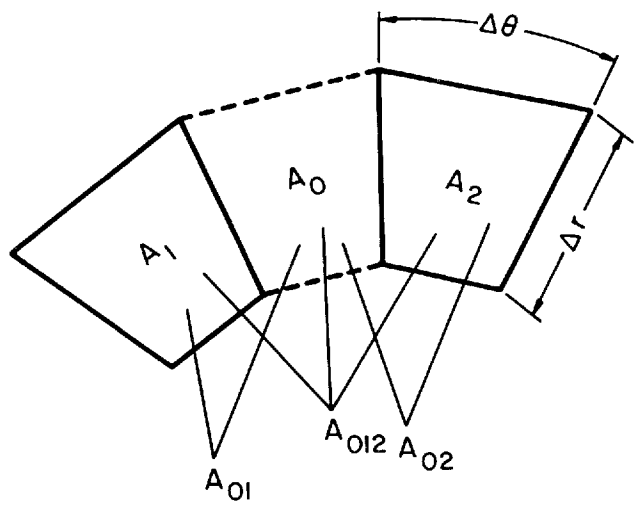

Figure 1. Finite areas used in Equation (6)

In a numerical method, such as finite volume method, the flow domain is divided into an appropriate number of finite volumes (i.e. areas in a two-dimensional domain), each of relatively small size. If there are n finite volumes, there will be n stochastic field values associate with these volumes. Let consider the fluctuating components $Y_i = Y(\mathbf{r}_i)$, $(i = 1, 2, \ldots, n)$ of the homogeneous stochastic field which is assumed to model the flow property variation around its expected value. The functions Y_i are random with mean zero, but correlated to each other. $\mathbf{r}_i$ represents the centroid location of a volume i. The correlation between fluctuations can be specified in terms of the covariance matrix C_{YY}, whose ij component is

$$C_{ij} = \operatorname{cov}[Y_i Y_j] = E[Y_i Y_j] = R(\Delta \mathbf{r}_{ij}), \tag{9}$$

where $\Delta \mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$ is the separation vector between the centroid of volume i and j, and R is the correlation function which has been discussed in previous paragraphs [equation (3)].

A stochastic field $Y = [Y_1, Y_2, \ldots, Y_n]^T$ can be generated

$$Y = L\,Z, \tag{10}$$

where $Z = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]^T$ is a vector consisting of n independent Gaussian random variables with mean zero and unit standard deviation. The lower triangular matrix L is obtained by the Cholesky decomposition of the covariance matrix C_{YY}. Hence,

$$E[ZZ^T] = I \quad \text{and} \quad L\,L^T = C_{YY} \tag{11}$$

where I is the identity matrix of appropriate dimensions. The superscript T denotes the transpose of the matrix. The vector Y generated by using equation (10) satisfies the original covariance matrix

$$E[YY^T] = E[LZ(LZ)^T] = L\,E[ZZ^T]\,L^T = C_{YY}. \tag{12}$$

Different vectors Y are easily obtained with the aid of equation (10) by generating different samples for the independent Gaussian random vector Z after the Cholesky decomposition is performed. Therefore, this technique can be conveniently used in Monte Carlo simulation.

The stochastic field Y can also be generated from C_{YY} by the spectral decomposition method. The spectral decomposition is especially useful when the Y's are highly correlated with each other. In this case, the Cholesky decomposition may become numerically difficult.[15] The eigenvalues and eigenvectors of covariance matrix C_{YY} are obtained by solving the following equation:

$$C_{YY}\,\phi_Y = \Lambda_Y\,\phi_Y, \tag{13}$$

where Λ_Y is a diagonal matrix consisting of the eigenvalues of C_{YY} along its diagonal, and $\phi_Y = [\phi_1, \phi_2, \ldots, \phi_n]^T$ is the model matrix whose column vector ϕ_i is the ith normalized eigenvector corresponding to the ith eigenvalue. A random independent Gaussian variable vector $Z = [\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n]$ with a zero mean and unit standard deviation can now be easily generated. Then the stochastic field can be obtained by the following expression:

$$Y = \phi_Y\,Z. \tag{14}$$

The autoregressive model

Consider a stochastic field in which there exist spatial structures. The spatial correlation of the field can be described by the covariances matrix C_{YY} which was discussed in the previous section. Let assume that the spatial structure of the stochastic field is represented by an autoregressive model.[10,11,16] The random values in the process are related through a simple linear equation expressing the dependence of the value at a point on the values at surrounding points. The autoregressive model is designed to model spatial variations in a statistically homogeneous

random field in which the stochastic dependence is local. The spatial dependence is quite different from a time series in which the structure can extend only in one direction, with the dependence only onto previously realized values. In space, the dependence is multidirectional and may have no directional preference. Therefore, it is impossible to generate values of the stochastic field sequentially from a set of known boundary values as it is in a time series model.

Let us consider a two-dimensional pipe cross-section, which is divided into n volumes [Figure 2(a)]. In the two-dimensional case, the first-order autoregressive model (nearest-point model) can be written as

$$Y_{ij}=\alpha_\theta(Y_{i-1,j}+Y_{i+1,j})+\alpha_r(Y_{i,j-1}+Y_{i,j+1})+\varepsilon_{i,j}, \tag{15a}$$

where

- $Y_{i,j}$ is the random variable satisfying the autoregressive relation,
- $\varepsilon_{i,j}$ is the normal random variable uncorrelated with other $\varepsilon_{i,j}$ and having a mean zero and unit standard deviation,
- α_θ is an autoregressive parameter expressing the degree of spatial dependence of $Y_{i,j}$ on its two neighbouring values in the θ direction, $Y_{i-1,i}$ and $Y_{i+1,j}$ ($|\alpha_\theta|<1$),
- α_r is a similar autoregressive parameter in the r direction, on its neighbouring values $Y_{i,j-1}$ and $Y_{i,j+1}$ ($|\alpha_r|<1$).

Figure 2(b) shows the notations used for $Y_{i,j}$ in the pipe flow domain. For the one-dimensional case, the autoregressive relation is

$$Y_i=\alpha(Y_{i-1}+Y_{i+1})+\varepsilon_i, \tag{16}$$

where the definition of the variables is similar to equation (15a).

In a statistically homogeneous medium, equation (15a) holds for every finite volume within the domain. If α_θ equals α_r, the medium has a statistically isotropic covariance structure. That is, the statistical dependence between neighbouring values is independent of the orientation of the vector separating those values.

Equation (15a) is written for an interior finite volume. For the boundary volumes, equation (15a) is used in a modified form. At a boundary of symmetry in the ith direction, we assume that the stochastic process from $(i-1, j)$ to (i, j) follows the same trend as that from (i, j) to $(i+1, j)$, i.e.

$$Y_{i-1,j}=Y_{i,j}-(Y_{i+1,j}-Y_{i,j}).$$

Equation (15) becomes

$$Y_{i,j}=C_Y\alpha_{rs}(Y_{i,j-1}+Y_{i,j+1})+C_Y\varepsilon_{i,j}, \tag{15b}$$

where the subscript s denotes symmetry boundary, and $C_Y=1/(1-2\alpha_{\theta s})$.

Since stochastic process does not exist beyond the wall, the process should have no dependence on any locations beyond the wall. Accordingly, equation (15a) becomes

$$Y_{i,j}=\alpha_{\theta w}(Y_{i-1,j}+Y_{i+1,j})+\alpha_{rw}(Y_{i,j-1})+\varepsilon_{i,j}, \tag{15c}$$

where the subscript w denotes wall boundary. The equations (15a)–(15e) form a system of n linear equations

$$Y=W\,Y+Z, \tag{17}$$

where Z is a random number vector

$$Z=[\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n], \tag{18}$$

and W is a spatial lag operator matrix.

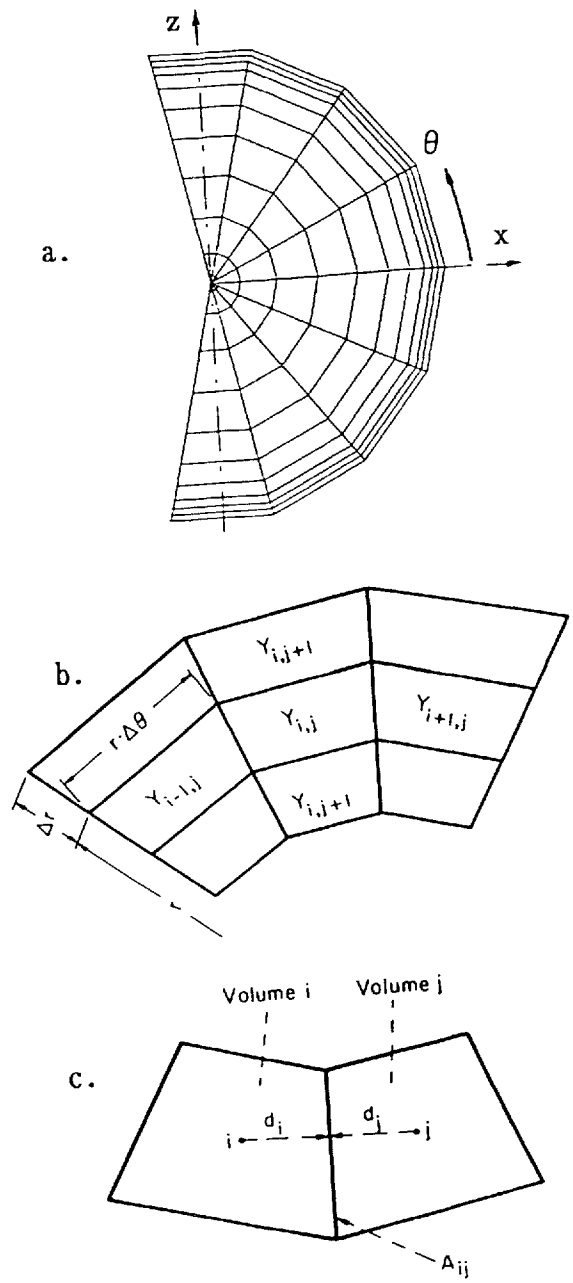

Figure 2. Computational domain: (a) mesh generated for a circular-pipe cross-section, (b) detail, (c) schematic diagram for the interpolation function

The calculation algorithm is the following. First, one generates n random Gaussian numbers, $\varepsilon_{i,j}$, with a mean zero. Then, the system equation (17) is solved for the value of $Y_{i,j}$, yielding an internally correlated sequence of random variables which satisfies the autoregressive relation. Because $\varepsilon_{i,j}$ has the mean equal to zero, so has Y. Any mean $\bar{\zeta}$ can be added to Y to obtain

ζ [equation (1)]. The process is normally distributed with a mean $\bar{\zeta}$ and a standard deviation σ_Y.

As indicated earlier, the vector Z has a mean zero and a unit standard deviation

$$E[Z]=\bar{\zeta}_Z=0, \tag{19}$$

$$\operatorname{cov}[Z\ Z^{\mathrm{T}}]=E[Z\ Z^{\mathrm{T}}]=I. \tag{20}$$

Equation (17) can be rewritten as

$$Y-WY=Z, \tag{21}$$

or

$$(I-W)\,Y=Z. \tag{22}$$

The vector Y can be obtained from (22)

$$Y=(I-W)^{-1}\,Z. \tag{23}$$

The matrix inversion $(I-W)^{-1}$ can be regarded as a filter operating on a random vector Z to yield an output vector Y with an internal correlation depending upon the parameters α_θ and α_r. At the beginning of the simulation, the matrix $(I-W)$ is to be inversed only once, the subsequent generation of the Y process can be easily done by simply multiplying the inversed matrix with a different Gaussian random vector Z.

The mean of Y is

$$E[Y]=E[(I-W)^{-1}\,Z]=(I-W)^{-1}\,E[Z], \tag{24}$$

because the components of W are constants for a statistically homogeneous medium. By applying equation (19), we obtain

$$E[Y]=0. \tag{25}$$

The covariance of the Y process is

$$\begin{aligned}\operatorname{cov}[Y]&=E[(Y-E[Y])(Y-E[Y])^{\mathrm{T}}],\\ &=E[Y\ Y^{\mathrm{T}}],\\ &=E[\{(I-W)^{-1}\,Z\}\{(I-W)^{-1}\,Z\}^{\mathrm{T}}],\\ &=\{(I-W)^{\mathrm{T}}(I-W)\}^{-1}\,E[Z\,Z^{\mathrm{T}}].\end{aligned} \tag{26}$$

By applying equation (20), with

$$S=\{(I-W)^{\mathrm{T}}(I-W)\}^{-1},$$

then

$$\operatorname{cov}[Y]=S\,I=S. \tag{27}$$

The correlation function matrix of the Y process is

$$R=\frac{1}{\sigma_Y^2}\operatorname{cov}[Y]=\frac{1}{\sigma_Y^2}\,S, \tag{28}$$

where σ_Y^2 is the variance of the statistically homogeneous Y process.

By specifying one finite volume for the origin of the correlation matrix, one can plot a series of one-dimensional correlation functions along vectors moving away from that volume. If the

process is isotropic, $\alpha_\theta = \alpha_r$, and all one-dimensional correlation functions will be equal. Generally, the vector oriented in the direction of the larger autoregressive parameter will have the slowest decaying correlation between neighbouring values. The integral scale η_θ or η_r can be calculated by integrating the one-dimensional correlation functions in the θ or r direction, respectively.

NUMERICAL ALGORITHM

Several deterministic numerical methods, such as finite differences, finite element and finite volume methods, are well-represented in computational fluid dynamics in recent years. The deterministic finite volume method for two-phase flow has been used in References 4 and 7, and it is adopted in this study. The basic flow equations were presented in previous work.[4,16] Here, a probabilistic approach is introduced by using the Neumann expansion in the framework of Monte Carlo method.

Probabilistic approach

In a flowing mixture of solids and fluid, the velocity, concentration and phase configurations are randomly distributed in space and time. The probabilistic approach can be used to evaluate both the mean and fluctuating quantities.

In a stochastic field, the coefficient matrix K of the system of governing equations is not unique. The matrix K can be divided into two parts: the mean part, and the fluctuation part due to the stochastic field. For complex problems such as two-phase flow computation, the direct Monte Carlo simulation is too expensive to be pursued at present time. A more efficient probabilistic approach is the Neumann expansion method, which is presented here.

The system of differential equations describing the flow is transformed into an algebraic system of equations according to the adopted numerical method (finite volume method in the present study). It can be written as

$$V = K^{-1} \cdot F, \tag{29}$$

where V is the unknown vector, K is a $n \times n$ coefficient matrix, and F is the right-hand-side vector. In our two-phase flow problem, the vector V takes successively the values of the velocity, concentration and eddy viscosity. Here

$$K = K_0 + \Delta K, \tag{30}$$

where K_0 is the mean part of K, and ΔK is the fluctuation part which contains the stochastic variables. K^{-1} can be expanded as follows:

$$\begin{aligned} K^{-1} &= (K_0 + \Delta K)^{-1}, \\ &= K_0^{-1} - (K_0^{-1})^2 \, \Delta K + (K_0^{-1})^3 \, \Delta K^2 - \ldots, \\ &= (I - P + P^2 - P^3 + \ldots) \, K_0^{-1}, \end{aligned} \tag{31}$$

where $P = K_0^{-1} \, \Delta K$. Substituting equation (31) into (29)

$$\begin{aligned} V &= K^{-1} \cdot F, \\ &= (I - P + P^2 - P^3 + \ldots) \, K_0^{-1} \cdot F, \\ &= V_0 - V_1 + V_2 - V_3 + \ldots, \end{aligned} \tag{32}$$

where

$$V_0 = K_0^{-1} F,$$

$$V_1 = P K_0^{-1} F, \quad \text{etc.}$$

This series of solution is equivalent to the following recursive equation:

$$K_0 \cdot V_i = \Delta K \cdot V_{i-1} \quad (i = 1, 2, \ldots). \tag{33}$$

Accordingly, the matrix K_0 has to be inverted only once in order to compute V. As an alternative, the solution for equation (33) can be obtained by initially decomposing the matrix K_0:

$$K_0 = L \cdot U, \tag{34}$$

where L and U are lower and upper triangular matrices, respectively. Then, the following equations with respect to the unknowns Q and V_0 are solved

$$L \cdot Q = F \quad \text{and} \quad U \cdot V_0 = Q. \tag{35}$$

The vectors Q and V_0 can be solved efficiently because of the matrices L and U are triangular. This algorithm (35) is used here to obtain V_i recursively.

The expansion series (32) may be terminated after a few terms if convergence of the series is confirmed by using the following criteria:

$$\frac{||V_i||}{\left|\left|\sum_{l=0}^{i} (-1)^l V_l\right|\right|} \leqslant \delta_{\text{err}}, \tag{36}$$

where δ_{err} is the specified error limit and $||V||$ is the normal length of vector V

$$||V|| = (V^{\text{T}} \cdot V)^{0.5}. \tag{37}$$

By solving (33) with (32), the vector V can be calculated. One notes that the stochastic field affects only ΔK, while K_0 is unchanged. The recalculation of equation (33) does not involve any matrix inversion or matrix factorization and, therefore, can be performed very efficiently. By repeating this procedure M times, the Monte Carlo simulation can then be proceeded.

$$E[V] = \frac{1}{M} \sum_{j=1}^{M} V_j, \tag{38}$$

$$\sigma_V^2[V] = \frac{1}{M} \sum_{j=1}^{M} \{(V_j - E[V]) \cdot (V_j - E[V])^{\text{T}}\}, \tag{39}$$

where V_j denotes the jth solution of equation (32). The outstanding feature of this approach of Monte Carlo simulation is that the matrix inversion or factorization is performed only once for all M solutions of unknown vector V. In our example for solid–liquid mixture flow computation in pipes, the vector V has three unknowns at a node: eddy viscosity, concentration and velocity.

The convergence criteria for the Neumann expansion series [equation (31)] is that the absolute values of all eigenvalues of $P = K_0^{-1} \cdot \Delta K$ are less than one. These criteria can be easily met no matter how large each component of the fluctuation matrix ΔK is in comparison with the corresponding component of K.

NUMERICAL EXPERIMENTS

The proposed computational method is illustrated for single fluid and liquid–solid-particle mixture pipe flow. The pipe radius is 2·57 cm, and the solid particles consist of sand grains of

0·17 mm diameter. The computational mesh in the pipe cross-section is symmetrical with respect to the vertical mid-plane [Figure 2(a)]. There are 14 radial sectors, and 10 circumferential rows in the computational domain.

Determination of the autoregressive parameter

As defined in equation (15a), the autoregressive parameter α expresses the degree of spatial dependence of the random process on its neighbouring points. The higher the value of α, the greater is the dependence. The value of α is a function of the integral scale of the stochastic field η and mesh spacing. Figure 3 shows the variation of the integral scale, normalized by the mesh spacing, as a function of α in the radial and circumferential directions, at $0{\cdot}128r$. As long as the integral scale is known, the corresponding α can be easily determined.

The integral scale has a definition and meaning similar to the mixing length scale in turbulence. By using an empirical expression of the mixing length for a circular pipe,[4] we have obtained the average of α over the pipe cross-section ($\alpha \approx 0{\cdot}5$).

Single-fluid pipe flow

A preliminary test was run for water flow in a circular pipe, where the random input was applied to the turbulent eddy viscosity. Two different stochastic models were employed: the one-dimensional autoregressive model and two-dimensional autoregressive model. The turbulent eddy viscosity can be written as

$$\nu_t = \bar{\nu}_t + \varepsilon, \tag{40}$$

where $\bar{\nu}_t$ represents the mean value. ε represents the stochastic part of ν_t, and can be obtained by the stochastic models for Y previously discussed in this paper. The input distribution of random number for the stochastic model is normal, with a mean of zero and a standard deviation of σ_ε.

In the case of using the one-dimensional stochastic model with velocity fluctuations only in the radial direction, the velocity in circumferential direction has no stochastic fluctuation. The

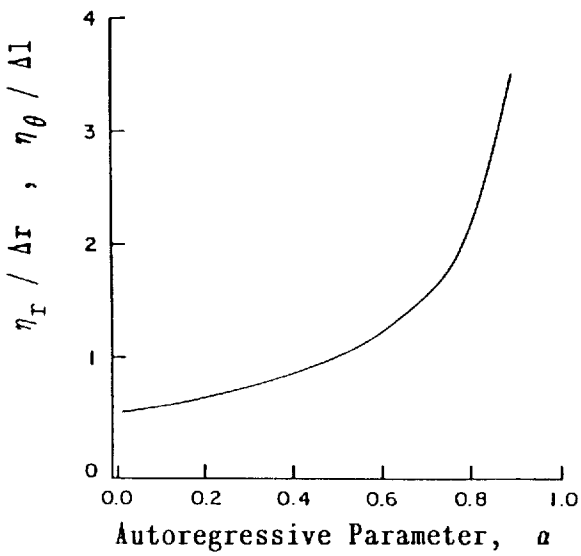

Figure 3. Variation of the integral scale η as a function of the autoregressive parameter α

velocity gradient at the wall decreases as the standard deviation of stochastic process increases, which implies the increase of the flow resistance. The velocity deviates from the deterministic value as a function of σ_ε of the stochastic process. With the increase of σ_ε, the velocity distribution tends to become flatter.

Figure 4 shows the computational results along the radius, at $\theta = 12{\cdot}9^\circ$ from the horizontal co-ordinate, by using a two-dimensional stochastic model for ν_t. In this case, the stochastic process is imposed on both r and θ directions. The fluid velocity is allowed to fluctuate in both directions. The deviations from the deterministic values show the importance of the probabilistic approach in the prediction of the mean velocity. The deviations increase with the increase of σ_ε, and they are more significant than in the one-dimensional stochastic process. The solid line in Figure 4 corresponds to the deterministic solution ($\sigma_\varepsilon = 0$), for which the empirical coefficients in the model are fitted to the experimental data. Nevertheless, the probabilistic solution reflects more realistically the intrinsic physics of the flow, and includes a supplementary parameter ($\sigma_\varepsilon > 0$) defining the flow characteristics. The probabilistic solution may better approximate the experiments for a larger variety of flow conditions after the adjustment of the empirical coefficients as in the deterministic solution.

Figure 5 shows the standard deviation of the computed velocity versus the input standard deviation for kinematic eddy viscosity, at one point (at $0{\cdot}742r$ and $\theta = 12{\cdot}9^\circ$). With the two-dimensional stochastic model, the computed velocity is less sensitive to the input standard deviation than in the case of the one-dimensional process.

With the normal distribution for ε (which is the input parameter for the stochastic process), the computed velocity has also a quasi-normal distribution. The probability of not rejecting the null hypothesis of normal distribution of computed velocity increases as the number of Monte Carlo simulation increases. Figure 6 shows the input normal distribution function for ε at $0{\cdot}742r$ and $\theta = 12{\cdot}9^\circ$.

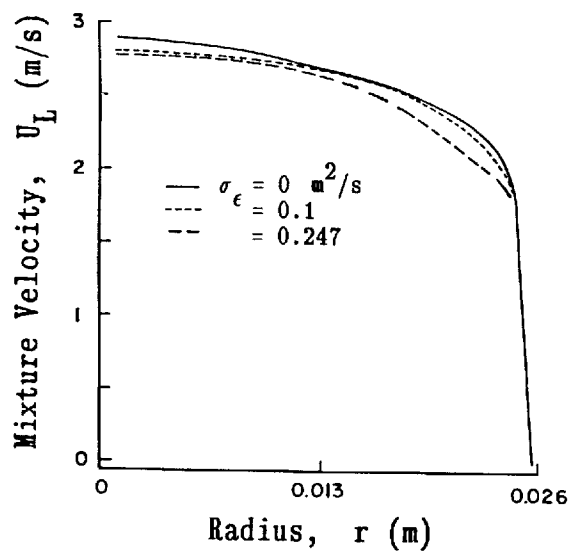

Figure 4. Single-fluid velocity prediction in the radial direction for different σ_ε by using the two-dimensional autoregressive model

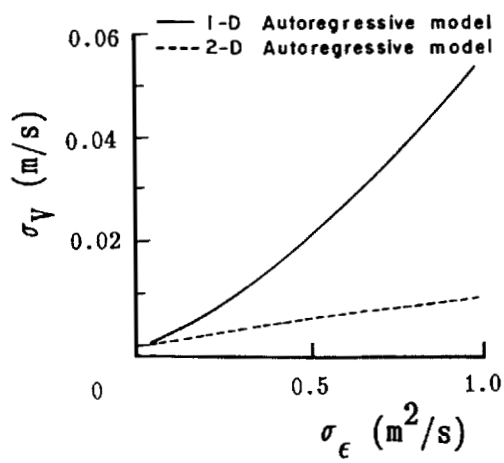

Figure 5. Correlation between the standard deviation of the computed mixture velocity (σ_V) and standard deviation of the turbulent eddy viscosity (σ_ϵ) at a point in single-fluid flow

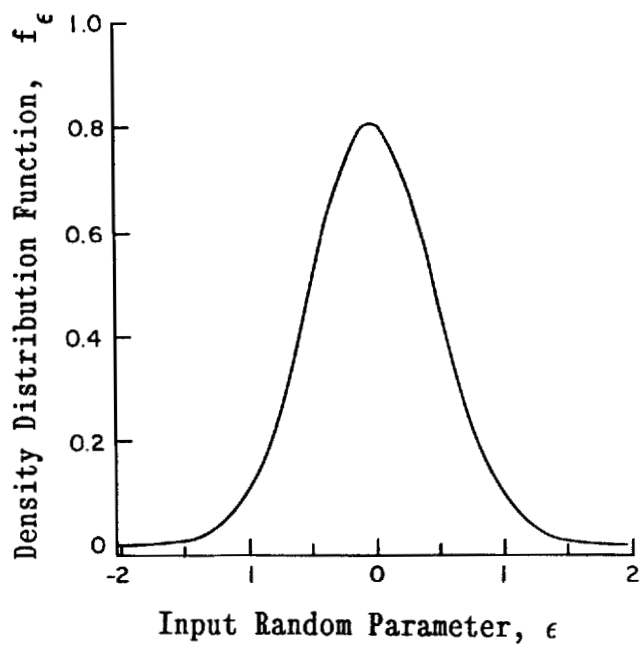

Figure 6. The density distribution function of the input random distribution for turbulent viscosity

Solid–liquid two-phase flow

The mixture consists of water at 20°C and 0·17 mm sand at an average concentration *in situ* of 25% by volume. The mean velocity over the pipe cross-section is 1·66 m s^{-1}. The corresponding mean value of the kinematic viscosity for water flow is $0{\cdot}18 \times 10^{-3}$ m^2 s^{-1}. Tests with a similar slurry have been presented in the previous work.[4,14]

A two-dimensional stochastic model with the turbulent viscosity as random input data was applied in simulations. Figure 7 shows the standard deviation σ_V of computed velocity as a function of the input standard deviation of the input eddy viscosity σ_ε at $0{\cdot}742r$ and $\theta = 12{\cdot}9°$. The two-dimensional stochastic model produced stable solutions. Another set of random input parameters could be related to the solid concentration fluctuations.

The computed velocity contours by the probabilistic method with $\sigma_\varepsilon = 0{\cdot}025$ is compared to the deterministic solution in Figure 8. No coefficient of the two-phase flow model was adjusted as compared to the deterministic model. The differences are due only to the change of the standard deviation of v_t fluctuations, from $\sigma_\varepsilon = 0$ to $\sigma_\varepsilon = 0{\cdot}025\ \mathrm{m^2\,s^{-1}}$. The Neumann expansion method converged with no more than 5 terms, with the accuracy of 0·5% between two successive

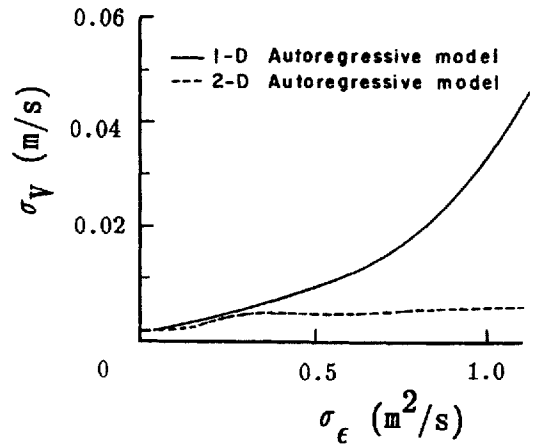

Figure 7. Correlation between the standard deviation of the computed mixture velocity (σ_V) and standard deviation of the turbulent eddy viscosity (σ_ε) at a point in solid–liquid mixture flow

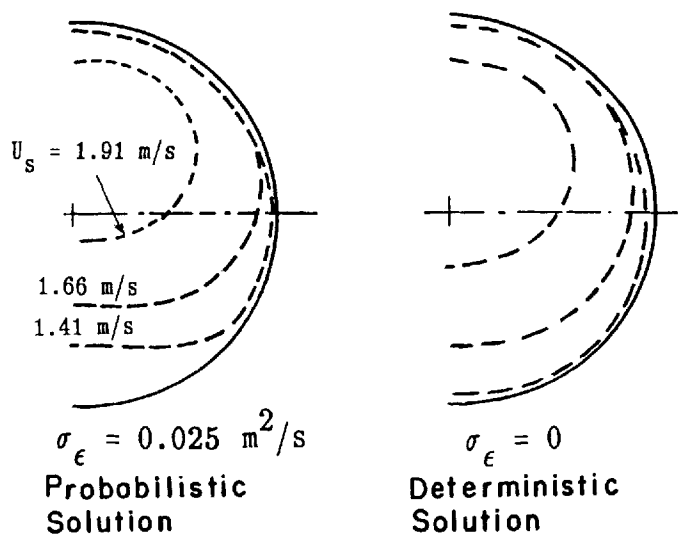

Figure 8. The velocity contours obtained by the probabilistic method for circular-pipe two-phase flow in comparison with the deterministic method

iterations [$\delta_{err} = 0{\cdot}005$ in equation (36)]. The CPU computational time has increased from 39·0 s for the deterministic solution to 63·4 s for the probabilistic solution, i.e. there is an increase of 62% of the CPU computational time. This, relatively small increase of the computational time, is due to the application of the Neumann method, where the main matrix is inversed only once, as it is in the deterministic approach. The probabilistic solution qualitatively agrees better with the experimental results because it shows a stratified flow at the pipe bottom.[4] The deterministic solution overpredicts the velocity of the flow in the lower half of the pipe considerably. The increase of the flow resistance at the pipe bottom is qualitatively better predicted with the probabilistic approach. This is the result of the non-linearity of the governing equations and of the spatial correlation considered for the fluctuating field. The objective of this paper was to describe the probabilistic field approach. Comparisons with experiments and other methods will be presented elsewhere.

The present study was performed with several simplifying assumptions. The random process was assumed homogeneous. The coefficient α was evaluated in average, from the mean value in the pipe cross-section of the turbulent mixing length. The fluctuating eddy viscosity, as well as other fluctuating input parameters such as solids concentrations, should be evaluated from or verified with experimental data.

CONCLUDING REMARKS

The probabilistic computational method for single- and two-phase flow described in this paper accounts for the stochastic character of the flow. The flow mechanisms can be more realistically reflected in the computational algorithm. The probabilistic governing equations were discretized in space using a finite volume method, and then solved by applying the Neumann expansion method. This method is time-efficient as compared to the direct Monte Carlo simulation, and its convergence can be guaranteed even for large fluctuations of the input parameters. The probabilistic approach can be used to evaluate both the mean and the fluctuating quantities. Computational results for liquid–solid particle mixture flow in a circular pipe illustrate the merit of the probabilistic approach.

The probabilistic numerical approach presented here can be applied in conjunction with other numerical methods (finite element, finite differences, or boundary element method) to solve partial differential equations. The approach is recommended to flows with large fluctuations of the field parameters and large spatial correlations of the fluctuations, such as flows with large turbulence scales and flow macrostructures. The construction of the mesh generation has to be related to the spatial correlations between fluctuations. In order to be more effective, the length scale of the mesh should be smaller than the integral length scale (η) of the fluctuating field. The probabilistic numerical approach is particularly useful in two-phase flow, were the phase concentration variations and particle–particle interactions are supplementary sources of fluctuations even in laminar flow.

ACKNOWLEDGEMENTS

This research was partially supported by the U.K. Centre for Computational Sciences and NSF.

APPENDIX I: NOMENCLATURE

A	area
$C_{Y,Y}$	covariance of the Y process ($=\mathrm{cov}[Y_i, Y_j]$)
F	right-hand-side vector in matrix equation

I	identity matrix
i, j	current nodes
K	coefficient matrix
n	total number of nodes
r	pipe radius
R	correlation function
$\mathbf{r}$	position vector
U_L and U_s	liquid and solid velocity, respectively
V	unknown vector in matrix equation
x, y, z	Cartesian co-ordinates
Y	stochastic field
Z	random number vector

Greek letters

α	autoregressive parameter
ε	random number of Gaussian distribution
γ	two-dimensional variance function
Λ_Y	eigenvalue matrix, defined in equation (13)
ν_t	kinematic turbulent eddy viscosity
η	integral scale of the fluctuating field
ρ	density
σ	standard deviation at a point
ζ	flow property

Superscripts

$[Y]^T$	transpose of matrix $[Y]$
$\bar{\zeta}$	mean value of ζ
$[Y]^{-1}$	inversion of matrix $[Y]$

Subscripts

i	i direction or ith node
i, j	interface (i, j)
x, z	co-ordinates defined in Figure 2(a)

APPENDIX II: DETERMINISTIC FINITE VOLUME METHOD

The system of governing differential equations for either velocity, concentration or eddy viscosity, is discretized by using a finite volume method (FVM). The resulted system of algebraic equations is solved for a deterministic (unic) set of coefficients.

In order to describe the FVM method adopted here, we consider a general transport equation for any scalar quantity ψ

$$\nabla\cdot(\psi\,\mathbf{U})-\nabla\cdot(\varepsilon_\psi\nabla\psi)-S_\psi=0, \tag{41}$$

where $\mathbf{U}$ is velocity vector transporting ψ by convection, ε_ψ is eddy diffusivity of ψ, and S_ψ is source term.

The transport equation is then integrated over a finite volume v_i

$$\int_{v_i}\nabla\cdot(\psi\,\mathbf{U})\,\mathrm{d}v_i-\int_{v_i}\nabla\cdot(\varepsilon_\psi\nabla\psi)\,\mathrm{d}v_i-\int_{v_i}S_\psi=0, \tag{42}$$

using Gauss's theorem, for any quantity f,

$$\int_v \nabla \cdot \mathbf{f}\, dv = \int_A \mathbf{n} \cdot \mathbf{f}\, dA,$$

$$\int_v f\, dv = \int_A f k_{sh} d\, dA.$$

The equation (42) is transformed into

$$\int_{A_i} \mathbf{n} \cdot (\psi \mathbf{U})\, dA - \int_{A_i} \mathbf{n} \cdot (\varepsilon_\psi \nabla \psi)\, dA - \int_{A_i} k_{sh} S_\psi d\, dA = 0, \tag{43}$$

where A_i is the control surface for ith control finite volume v_i, $\mathbf{n}$ is outward unit normal, d is the distance from the node i to the control surface, and k_{sh} is the coefficient related to the typical shape of the finite volume.

The surface integral equation (43) can be approximated to an algebraic equation

$$\sum_j \mathbf{n}_{i,j} \cdot (\psi \mathbf{U})_{i,j} A_{i,j} - \sum_j \mathbf{n}_{i,j} \cdot (\varepsilon_\psi \nabla \psi)_{i,j} A_{i,j} - \sum_j (S_\psi)_{i,j} d_{i,j} k_{sh} = 0, \tag{44}$$

where $A_{i,j}$ is the area of the interface (i, j), $\mathbf{n}_{i,j}$ is the unit normal on the interface (i, j), $(\cdot)_{i,j}$ is the quantity $(\cdot)$ on the interface (i, j), and $d_{i,j}$ is the distance from the node i to the interface (i, j).

Any quantity on interface (i, j) can be interpolated between the values at each node on either side of that interface (Figure 2). The interpolation can be performed algebraically with a linear expression

$$\psi_{i,j} = \frac{\psi_i d_j + \psi_j d_i}{d_i + d_j}, \tag{45}$$

or analytically if the upwind effect is important (i.e. $\mathbf{U}_i \mathbf{n}_{i,j} d_i / \xi_j$ or/and $\mathbf{U}_j \mathbf{n}_{i,j} d_j / \xi_j$ are larger than unity). In this last situation, the velocities and diffusion coefficients at nodes i and j are included in the interpolation function.

Equation (44) is written for each finite volume in the flow domain. In our simulation of uniform-pipe flow, each finite volume has a base defined by the mesh generation in the pipe cross-section [Figure 2(a)], and a height along the pipe axis equal to unity. By using interpolation formulas of type (45), all the interfacial values are expressed as a function of the node values. There are total n algebraic equations for the flow domain which is divided into n finite volumes. This system of n algebraic equations can be written as a matrix equation

$$KV = F, \tag{46}$$

where K is a $n \times n$ coefficient matrix, V is the unknown vector $V = [\psi_1, \psi_2, \ldots, \psi_n]^T$, and F is the right-hand-side vector.

Equation (46) is then solved for the unknown vector V by a Gaussian elimination or iteration scheme. If experimental data are available as reference, the vector V has to be compared to the mean value of the corresponding instantaneous measurements.

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